HG(3) — M (8) Sc. & Arts. (Func. Anal.)

2015

Time : 3 hours

Full Marks : 70

Pass Marks : 32

Candidates are required to give their answers in their own words as far as practicable.

The questions are of equal value.

Answer any five questions.

1. (a) Define Banach space and give an example of a normed linear space which is not a Banach space.

   (b) Let p be a real number such that $1 \le p < \infty$ and ℓ_p be the set of all infinite sequences $x = (x_1, x_2, ..x_n, ...)$ of scalars in which the infinite series $\sum_{i=1}^{\infty} |x_i|^p < \infty$. For $x = (x_i) \in \ell_p$

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and $y = (y_i) \in \ell_p$, we define $x + y = (x_i + y_i)$ and $\lambda x = (\lambda x_i)$ and $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$, then prove that ℓ_p is a Banach space.

2. (a) Let M be closed linear subspace of a normed linear space E. If Norm of a Coset $[x] = x + m$ in the quotient space E/M is defined by :

$$\|x + M\| = \text{Inf}\{\|x + v\| : v \in M\}$$

Then prove that E/M is a normed linear space. Further if E is a Banach space, then prove that E/M is also a Banach space.

3. State and prove Hahn-Banach extension theorem on a normed linear space.

4. (a) Define continuous linear functional on a normed linear space with an example.

   (b) A linear transformation T from a normed linear space E into a normed linear space F

Contd.

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is continuous iff the image T(s) of closed unit sphere $S = \{x \in E : \|x\| \le 1\}$ of E is a bounded set in F.

5. (a) Prove that L^p-space is a mormed linear space.

(b) In a normed linear space E, prove that $|\|x\| - \|y\|| \le \|x - y\| \ \forall\, x, y \in E$ and also prove that norm function is a continuous function.

6. (a) Define Inner Product space and give an example of an incomplete inner product space.

(b) Prove that every Inner Product space E is a normed linear space with respect to the norm defined by inner product as

$$\|x\| = +\,(x,x)^{\frac{1}{2}} = +\sqrt{(x,x)}\ \forall\, x \in E.$$

7. If E and F are normed linear spaces over a field K then the set B(E, F) of all continuous (i.e. bounded) linear transformations of E into F is

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it-self a normed linear space with respect to pointwise linear operations and the norm defined by $\|T\| = \sup_{0 \ne x \in E} \frac{\|T(x)\|}{\|x\|}$, $T \in B(E, F)$.

Moreover if F is a Banach space, then B(E, F) is also a Banach space.

8. (a) State and prove parallelogram law in a Hilbert space.

(b) Give an example of a Banach space which is not a Hilbert space.

9. State and prove Bessel's inequality in a Hilbert space.

10. (a) State and prove Lemma of F. Riesz on closed convex set in a Hilbert space.

(b) If M and N are closed linear subspaces of a Hilbert space H such that M ⊥ N then linear subspace N + M is also closed.

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