HG(3) — Math (6)
New (Sc. / Arts)

2019

Time : 3 hours

Full Marks : 90

Pass Marks : 41

Candidates are required to give their answers in their own words as far as practicable.

The questions are of equal value.

Answer any ***six*** *questions.*

1. (a) Define an automorphism on a group G. Prove that a mapping f defined on a group G by $f(x) = x^{-1}$, for all $x \in G$ is an automorphism on G if and only if G is abelian.

   (b) Introduce the concept of an inner automorphism of a group. Prove that the set of all inner automorphisms of a group G is a normal subgroup of the group of all automorphisms of G.

2. (a) Define the centre Z of a group G and prove that if G/Z is cyclic then G is abelian.

   (b) Let p be a prime number then G a group of order p^2. Prove that G is abelian.

3. (a) State and prove Cauchy's theorem for finite abelian groups.

   (b) If H is a p-Sylow subgroup of a group G and $x \in G$, then prove that $x^{-1} H x$ is also a p-Sylow subgroup of G.

4. (a) Introduce the concept of an ideal in a ring. If R is a commutative ring with unity element 1 and if $a_o \in R$ then prove that $a_o R = \{a_o \cdot r \mid r \in R\}$ is an ideal in R.

   (b) State and prove divison algorithm for a polynomial ring F[x] over a field F.

5. Define the quotient ring R/I of a given ring R with respect to a given ideal I of R. Prove that if R is commutative then so is R/I and if R has a unity element 1 and I is a proper ideal then R/I has a unity element.

6. Define a Unique Factorisation Domain and prove that every Euclidean Domain is a Unique Factorisation Domain.

7. (a) Show that the set D[0, 1] of all real valued differentiable functions on [0, 1] is a real vector space under pointwise linear operations on D[0, 1].

(b) Let V be a vector space of dimension n. Prove that any set of n linearly independent elements of V is a basis of V.

8. (a) If a vector space V over a field F has dimension n with n > 0, then prove that V is isomorphic to the vector space $V_n(F)$ of all n-tuples of scalars.

(b) Prove that the vectors (x_1, x_2) and (y_1, y_2) in $V_2(F)$ are linearly dependent if and only if $x_1y_2 - x_2y_1 = 0$.

9. Let V and V′ be vector spaces over a field F. If dim V = n and $T : V \to V'$ is a linear transformation of rank r then prove that T has nullity (n – r).

10. (a) Define Eigen Values and Eigen vectors of a linear operator T on a finite dimensional vector space. Prove that the eigen vectors of T belonging to different eigen values of T are linearly independent.

(b) Find the eigen values of the matrix

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}.$$

11. (a) State and prove Cauchy – Schwarz inequality in a Hilbert space.

(b) Construct a Banach space which is not a Hilbert space.

12. Introduce the concept of sub-module of a given module. If A and B are sub-modules of a module M then prove that :

(a) $A \cap B$ is a sub-module of M.

(b) $A + B = \{a + b \mid a \in A, b \in B\}$ is a sub-module of M.

(c) $(A + B)/B$ is isomorphic to $B/(A \cap B)$.