HG(3) — Math (5)
Sc. & Arts

2020

Time : 3 hours

Full Marks : 90

Pass Marks : 41

Candidates are required to give their answers in their own words as far as practicable.

The questions are of equal value.

Answer any six questions.

1. (a) Let $f \in R[a, b]$ and let m, M be the bounds of f on [a, b], than prove :

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \text{ if } b \geq a \text{ and}$$

$$m(b-a) \geq \int_a^b f(x)dx \geq M(b-a) \text{ if } b \leq a.$$

(b) Show that if f is defined on [a, b] by $f(x) = k, \forall x \in [a, b]$, where k is a constant.

this f is R - integrable on [a, b]

and $\int_a^b k dx = k(b-a)$

2. (a) If f is continuous on [a, b] then prove that it is R-integrable on [a, b].

(b) Give an example of a bounded function which is not R-integrable.

3. (a) State Dirichlet's test for convergence of an improper integrals and hence test the convergence of the integral :

$$\int_a^\infty \frac{1}{\sqrt{x}} \sin x \, dx, \ a > 0$$

(b) Test the convergence of the integral $\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx$, where m and n are positive integers.

4. State and prove Implicit function theorem.

5. (a) Obtain Cauchy-Riemann equations in polar form.

(b) Prove : $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$.

6. (a) Prove that at each point z of a domain where f(z) is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

(b) Find the fixed points and normal form of the bilinear transformation $w = \frac{z}{z-2}$.

7. (a) Prove that every bilinear transformation maps circles or straight lines into circles and straight lines.

(b) Find the bilinear transformation that maps the points $z_1 = \infty, z_2 = i, z_3 = 0$ into the points $w_1 = 0, w_2 = i, w_3 = \infty$.

8. (a) Prove that continuity is necessary but not sufficient condition for the existence of a finite derivative of analytic function.

(b) Show that the function $f(z) = \bar{z}$ is not differentiable at any point.

9. (a) Define metric space with suitable example. Given any three points x, y, z in a metric space (X, d).

Prove : $|d(x, z) - d(y, z)| \leq d(x, y)$.

(b) Prove that the mapping $d : R^2 \times R^2 \to R$ defined by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_1), y = (y_1, y_2) \in R^2$ is a metric on R^2.

10. (a) Define Cauchy sequence and prove that every convergent sequence $\{x_n\}$ of points of a metric space (E, d) is a Cauchy sequence, but the converse is not true in general.

(b) Prove that an open sphere in a metric space (B, d) is an open set.

11. (a) Define complete metric space. Prove that a subspace Y of a complete metric space (X, d) is complete if and only if Y is closed.

(b) State and prove Cantor's intersection theorem.

12. Define a topological spapce with an example. Prove that in a topological space (X, τ), an arbitrary intersection of closed sets is closed and finite union of closed sets is closed.